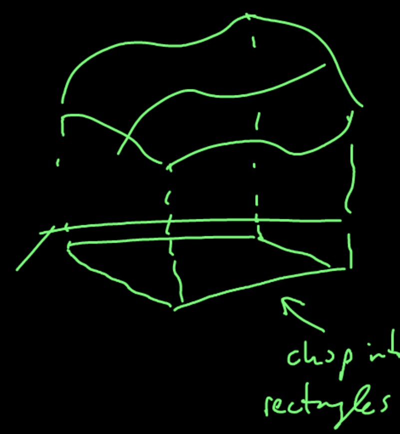
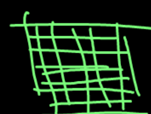
Double Integrals over Rectangles

Suppose $f : R \to \mathbb{R}$ is a function with a rectangular domain. By way of analogy to the integral in Calculus I:

	Calculus I	Calculus III
Symbol:	$\int_{t=a}^{b} f(t)\,dt$	$\iint_R f(x,y)\,dA$
Geometry:	Net Area ←	→ Net Volume
Formalism:	limit of sums of signed areas ←	→ limit of sums of signed volumes



NB: Pictures are essentially the same as in Calculus I...

Defn: The integral of $f : R \to \mathbb{R}$ for rectangle 
$R = [a,b] \times [c,d]$ is $\iint_R f(x,y)\,dA := \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y.$

NB: This definition is not perfect mathematically, but at this level we won't really work with the definition anyhow...

Q: How will we compute "double integrals", if not by definition?

A: In practice, we will use Fubini's Theorem, which relates the double integral to "iterated integrals."

Prop (Fubini's Theorem): If $f : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then

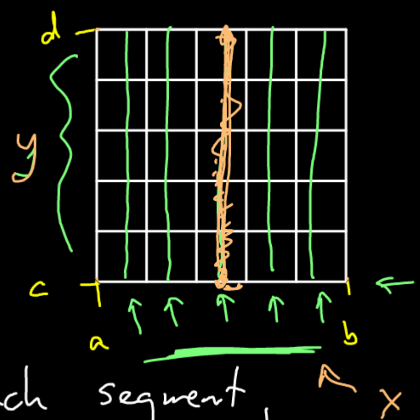
$$\iint_R f(x,y)\,dA = \int_{y=c}^{d}\left(\int_{x=a}^{b} f(x,y)\,dx\right)dy = \int_{x=a}^{b}\left(\int_{y=c}^{d} f(x,y)\,dy\right)dx.$$

Idea: $R = [a,b] \times [c,d]$ can be cut into smaller rectangles in two different ways

① Chop R vertically first, then horizontally. ←

② Chop R horizontally first, then vertically.

Picture



When we chop one direction and integrate along each segment, we measure the contribution of f "in sheets", which is naturally interpretted as iterating integrals to compute the double integral.

NB: The above idea is as close as we'll get to a proof of Fubini's Theorem... It's hard to prove, and we would need a much better formal definition of the integral !!

Ex: Compute $\iint_R (x - 3y^2)\,dA$ where $R = [0,2] \times [1,2]$.

Sol:

$$\iint_R (x-3y^2)\,dA = \int_{y=1}^{2}\int_{x=0}^{2} (x-3y^2)\,dx\,dy$$

$$= \int_{y=1}^{2} \left[\tfrac{1}{2}x^2 - 3xy^2\right]_0^2 dy$$

$$= \int_{y=1}^{2} (2 - 6y^2)\,dy$$

$$= \left[2y - 2y^3\right]_{y=1}^{2}$$

$$= (4 - 2\cdot 8) - (2-2)$$

$$= 4 - 16 - 0$$

$$= -12$$

□

Exercise: Evaluate first wrt y and then wrt x.

Ex: Compute $\iint_R f(x,y)\,dA$ for each below.

① $f(x,y) = \dfrac{xy^2}{x^2+1}$, $R = [0,1]\times[-3,3]$.

Sol: $\iint_R \dfrac{xy^2}{x^2+1}\,dA$

$$= \int_{x=0}^{1}\int_{y=-3}^{3} \frac{xy^2}{x^2+1}\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{x}{x^2+1}\int_{y=-3}^{3} y^2\,dy\,dx$$

$$= \int_{x=0}^{1} \frac{x}{x^2+1}\left[\frac{1}{3}y^3\right]_{y=-3}^{3} dx$$

$$= \left(\frac{1}{3}3^3 - \frac{1}{3}(-3)^3\right)\int_{x=0}^{1}\frac{x}{x^2+1}\,dx$$

$$= (9-(-9))\int_{x=0}^{1}\frac{1}{2}\cdot\frac{2x}{x^2+1}\,dx$$

$\left(\begin{array}{ll} u = x^2+1 & du = 2x\,dx \\ u(0) = 1 & u(1) = 2 \end{array}\right.$

$$= \frac{18}{2}\int_{u=1}^{2}\frac{1}{u}\,du$$

$$= 9\left[\ln|u|\right]_{u=1}^{2}$$

$$= 9(\ln(2) - \ln(1))$$

$$= 9\ln(2).$$

② $f(x,y) = \dfrac{x}{1+xy}$, $R = [0,1]\times[0,1]$

Sol: $\iint_R f(x,y)\,dA = \int_{y=0}^{1}\int_{x=0}^{1}\dfrac{x}{1+xy}\,dx\,dy$ ← hard!

$$= \int_{x=0}^{1}\int_{y=0}^{1}\frac{x}{1+xy}\,dy\,dx$$

$\left(\begin{array}{ll} u(y) = 1+xy & du = x\,dy \\ u(0) = 1 & u(1) = 1+x \end{array}\right.$

$$= \int_{x=0}^{1}\int_{u=1}^{1+x}\frac{1}{u}\,du\,dx$$

$$= \int_{x=0}^{1}\left[\ln|u|\right]_{u=1}^{1+x} dx$$

$$= \int_{x=0}^{1}(\ln(1+x) - \ln(1))\,dx$$

$$= \int_{x=0}^{1}\ln(1+x)\,dx$$

$\left(\begin{array}{ll} w = \ln(1+x) & dv = dx \\ dw = \frac{1}{1+x}dx & v = x \end{array}\right.$

$$= \left[x\ln(1+x)\right]_{x=0}^{1} - \int_{x=0}^{1}\frac{x}{1+x}\,dx$$

$$= \ln(2) - \int_{x=0}^{1}\left(1 - \frac{1}{1+x}\right)dx$$

$$= \ln(2) - \left[x - \ln|1+x|\right]_{x=0}^{1}$$

$$= \ln(2) - ((1-\ln(2)) - 0)$$

$$= 2\ln(2) - 1$$

Ex: Compute $\iint_R y\sin(xy)\,dA$ for $R=[1,2]\times[0,\pi]$.

Sol: $$\iint_R y\sin(xy)\,dA = \int_{x=1}^{2}\int_{y=0}^{\pi} y\sin(xy)\,dy\,dx \leftarrow \text{hard}$$

$$= \int_{y=0}^{\pi}\int_{x=1}^{2} y\sin(xy)\,dx\,dy \qquad \left(\begin{array}{ll} u(x)=xy & du = y\,dx \\ u(1)=y & u(2)=2y \end{array} \right.$$

$$= \int_{y=0}^{\pi}\int_{u=y}^{2y} \sin(u)\,du\,dy$$

$$= \int_{y=0}^{\pi} \Big[-\cos(u)\Big]_{u=y}^{2y}\,dy$$

$$= \int_{y=0}^{\pi} \big(-\cos(2y)+\cos(y)\big)\,dy$$

$$= \left[-\frac{1}{2}\sin(2y)+\sin(y)\right]_{y=0}^{\pi}$$

$$= \left(-\frac{1}{2}\sin(2\pi)+\sin(\pi)\right) - \left(-\frac{1}{2}\sin(0)+\sin(0)\right)$$

$$= \left(-\frac{1}{2}\cdot 0 + 0\right) - (0+0)$$

$$= 0$$

$\blacksquare$

Average Value

Defn: The <u>average value</u> of function $f(x,y)$ on region R is

$$f_{avg} := \frac{1}{\text{area}(R)} \iint_R f(x,y)\, dA.$$

Ex: Compute the average value of $f(x,y)$ on R.

① $f(x,y) = x^2 y$, $\quad R = [-1,1] \times [0,5]$.

Sol:

$$f_{avg} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$

$$= \frac{1}{(1-(-1))(5-0)} \int_{x=-1}^{1} \int_{y=0}^{5} x^2 y \, dy\, dx$$

$$= \frac{1}{2 \cdot 5} \int_{x=-1}^{1} \left[\tfrac{1}{2} x^2 y^2\right]_{y=0}^{5} dx = \frac{1}{10} \cdot \frac{1}{2} \int_{x=-1}^{1} 25x^2\, dx$$

$$= \frac{25}{20} \left[\tfrac{1}{3} x^3\right]_{x=-1}^{1} = \frac{5}{4} \cdot \frac{1}{3} \left(1^3 - (-1)^3\right) = \frac{5}{4} \cdot \frac{1}{3} \cdot 2 = \frac{5}{6} \quad \boxed{//}$$

② $f(x,y) = \dfrac{xy}{1+x^4}$, $\quad R = [-1, \sqrt[4]{3}] \times [-2, \sqrt{2}]$.

Sol:

$$f_{avg} = \frac{1}{\text{area}(R)} \iint_R \frac{xy}{1+x^4}\, dA$$

$$= \frac{1}{(3^{1/4}+1)(\sqrt{2}+2)} \int_{x=-1}^{3^{1/4}} \int_{y=-2}^{\sqrt{2}} \frac{xy}{1+x^4}\, dy\, dx$$

$$= \frac{1}{(3^{1/4}+1)(2+\sqrt{2})} \int_{x=-1}^{3^{1/4}} \frac{1}{2} \cdot \frac{x}{1+x^4} \left[y^2\right]_{y=-2}^{\sqrt{2}} dx$$

$$= \frac{1}{(3^{1/4}+1)(2+\sqrt{2})} \int_{x=-1}^{3^{1/4}} \frac{x}{1+x^4} \frac{1}{2}(2-4)\, dx$$

$$= -\frac{2}{(3^{1/4}+1)(2+\sqrt{2})} \int_{x=-1}^{3^{1/4}} \frac{1}{2} \frac{2x}{1+(x^2)^2}\, dx \qquad \begin{pmatrix} u = x^2 & du = 2x\,dx \\ u(-1) = 1 & u(3^{1/4}) = \sqrt{3} \end{pmatrix}$$

$$= -\frac{2}{(3^{1/4}+1)(2+\sqrt{2})} \frac{1}{2} \int_{u=1}^{\sqrt{3}} \frac{1}{1+u^2}\, du$$

$$= -\frac{1}{(3^{1/4}+1)(2+\sqrt{2})} \left[\arctan(u)\right]_{u=1}^{\sqrt{3}}$$

$$= -\frac{\arctan(\sqrt{3}) - \arctan(1)}{(3^{1/4}+1)(2+\sqrt{2})} = \frac{-\frac{\pi}{3} + \frac{\pi}{4}}{(3^{1/4}+1)(2+\sqrt{2})} = -\frac{\pi}{12(3^{1/4}+1)(2+\sqrt{2})} \quad \boxed{//}$$

NB:

$$\begin{cases} \arctan(\sqrt{3}) = \frac{\pi}{3} \\ \arctan(1) = \frac{\pi}{4} \end{cases}$$

Double Integrals over General Regions

If the region of integration is not rectangular, we can still compute the integral using similar ideas to those above.

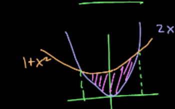
<u>Idea</u>: If the boundary of the region of integration can be expressed as a <u>union of continuous curves</u> in x or y, then we can still integrate in slices over the region. Thus we can turn $\iint_R f(x,y)\,dA$ into an iterated integral!

<u>Ex</u>: Compute $\iint_R (x+2y)\,dA$ where R is the region bounded by the curves $y=2x^2$ and $y=1+x^2$.

<u>Sol</u>: $y=1+x^2$ intersects $y=2x^2$ when

$$1+x^2=2x^2 \iff 0=x^2-1 \iff 0=(x+1)(x-1)$$
$$\iff x=-1 \text{ OR } x=1$$

Picture



$$R=\left\{(x,y): \begin{matrix} 2x^2 \le y \le 1+x^2, \\ -1 \le x \le 1 \end{matrix}\right\}$$

$$\therefore \iint_R (x+2y)\,dA = \int_{x=-1}^{1}\int_{y=2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{x=-1}^{1}\left[xy+y^2\right]_{y=2x^2}^{1+x^2} dx = \int_{x=-1}^{1}\left(\left(x(1+x^2)+(1+x^2)^2\right)-\left(x(2x^2)+(2x^2)^2\right)\right)dx$$

$$= \int_{x=-1}^{1}\left(x+x^3+1+2x^2+x^4-2x^3-4x^4\right)dx = \int_{x=-1}^{1}\left(-3x^4-x^3+2x^2+x+1\right)dx$$

$$= \left[-\tfrac{3}{5}x^5-\tfrac{1}{4}x^4+\tfrac{2}{3}x^3+\tfrac{1}{2}x^2+x\right]_{x=-1}^{1} = \left(-\tfrac{3}{5}-\tfrac{1}{4}+\tfrac{2}{3}+\tfrac{1}{2}+1\right)-\left(\tfrac{3}{5}-\tfrac{1}{4}-\tfrac{2}{3}+\tfrac{1}{2}-1\right)$$

$$= -2\cdot\tfrac{3}{5}+2\cdot\tfrac{2}{3}+2\cdot 1 = 2\left(-\tfrac{9}{15}+\tfrac{10}{15}+\tfrac{15}{15}\right)=\tfrac{32}{15}$$

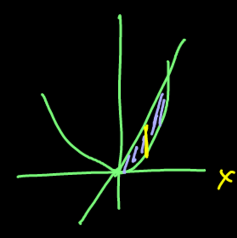
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Ex: Compute $\iint_R f(x,y)\,dA$ for ...

① $f(x,y) = x^2+y^2$ on R the region bounded by $y=2x$ and $y=x^2$.

Sol: $2x = x^2 \iff x(x-2)=0 \iff x=0$ OR $x=2$

Picture



$\therefore R = \{(x,y) : 0 \le x \le 2,\ x^2 \le y \le 2x\}$.

$\therefore \iint_R (x^2+y^2)\,dA = \int_{x=0}^{2}\int_{y=x^2}^{2x} (x^2+y^2)\,dy\,dx$

$= \int_{x=0}^{2} \left[x^2 y + \frac{1}{3}y^3\right]_{y=x^2}^{2x} dx = \int_{x=0}^{2} \left(\left(2x^3 + \frac{8}{3}x^3\right) - \left(x^4 + \frac{1}{3}x^6\right)\right) dx$

$= \int_{x=0}^{2} \left(\frac{14}{3}x^3 - x^4 - \frac{1}{3}x^6\right) dx = \left[\frac{7}{6}x^4 - \frac{1}{5}x^5 - \frac{1}{21}x^7\right]_{x=0}^{2}$

$= \left(\frac{7}{6}\cdot 16 - \frac{1}{5}\cdot 32 - \frac{1}{21}\cdot 128\right) - 0 = \frac{56}{3} - \frac{32}{5} - \frac{128}{21}$ ▨

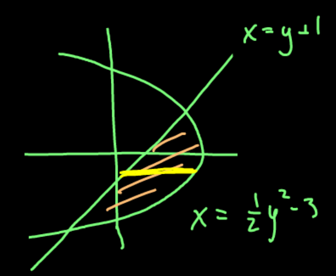
Ps. Good enough...

② $f(x,y) = xy$ on R the region bounded by $y = x-1$ and $y^2 = 2x+6$.

Sol: $\iint_R xy\,dA$

$y = x-1$ intersects $y^2 = 2x+6$ when

Picture



$\frac{1}{2}y^2 - 3 = x = y+1 \iff y^2 - 2y - 8 = 0 \iff (y-4)(y+2) = 0 \iff y = 4$ OR $y = -2$

$\therefore R = \{(x,y) : -2 \le y \le 4,\ y+1 \le x \le \frac{1}{2}y^2 - 3\}$

$\therefore \iint_R xy\,dA = \int_{y=-2}^{4}\int_{x=y+1}^{\frac{1}{2}y^2-3} xy\,dx\,dy$

$= \int_{y=-2}^{4} y\left[\frac{1}{2}x^2\right]_{x=y+1}^{\frac{1}{2}y^2-3} dy = \frac{1}{2}\int_{y=-2}^{4} y\left(\left(\frac{1}{2}y^2-3\right)^2 - (y+1)^2\right) dy$

$= \frac{1}{2}\int_{y=-2}^{4} y\left(\frac{1}{4}y^4 - 3y^2 + 9 - y^2 - 2y - 1\right) dy = \frac{1}{2}\int_{y=-2}^{4} \left(\frac{1}{4}y^4 - 4y^2 - 2y + 8\right) dy$

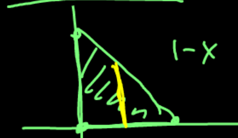
$= \frac{1}{2}\left[\frac{1}{20}y^5 - \frac{4}{3}y^3 - y^2 + 8y\right]_{y=-2}^{4} = \frac{1}{2}\left(\left(\frac{4^5}{20} - \frac{4^4}{3} - 16 + 32\right) - \left(-\frac{32}{20} + \frac{32}{3} - 4 - 16\right)\right)$

$= \frac{1}{2}\left(\frac{4^4}{5} - \frac{4^4}{3} + 16 + \frac{8}{5} - \frac{32}{3} + 20\right) = \frac{128+4}{5} - \frac{128+16}{3} + 18 = \frac{132}{5} - \frac{144}{3} + 18$

$= \frac{132}{5} - 48 + 18 = \frac{132}{5} - 30 = \frac{132-150}{5} = -\frac{18}{5}$ ▨

③ $f(x,y) = 1-x-y$ on R the triangle w/ vertices $(0,0)$, $(1,0)$, $(0,1)$.

Picture



$\therefore R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1-x\}$

Sol: $\iint_R (1-x-y)\, dA$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_{x=0}^{1} \left[y - xy - \tfrac{1}{2}y^2 \right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \left(1-x - x + x^2 - \tfrac{1}{2} + x - \tfrac{1}{2}x^2 - 0 \right) dx$$

$$= \int_{x=0}^{1} \left(\tfrac{1}{2} - x + \tfrac{1}{2}x^2 \right) dx$$

$$= \left[\tfrac{1}{2}x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 \right]_{x=0}^{1}$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6} \right) - (0 - 0 + 0)$$

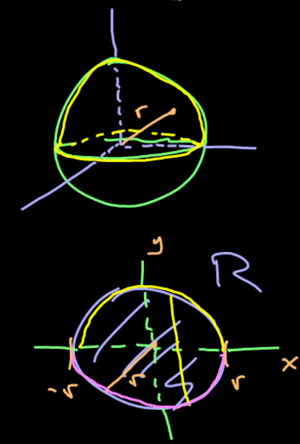
$$= \frac{1}{6}$$

▨

Ex: Compute the volume of the sphere of radius r.

Sol: ① Compute $f(x,y)$ ...

Picture



NB: Vol(upper hemisphere) = Vol(lower hemisphere).

$\therefore$ Vol(sphere) = 2 Vol (upper hemisphere).

$$x^2+y^2+z^2=r^2 \iff z=\pm\sqrt{r^2-x^2-y^2}$$

$$\therefore \quad f(x,y)=\sqrt{r^2-x^2-y^2}$$

② Parametrize R.

$$R=\{(x,y): x^2+y^2\le r^2\}=\{(x,y): -r\le x\le r, -\sqrt{r^2-x^2}\le y\le \sqrt{r^2-x^2}\}$$

③ Solve $\iint_R \sqrt{r^2-x^2-y^2}\,dA=\int_{x=-r}^{r}\int_{y=-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\sqrt{r^2-x^2-y^2}\,dy\,dx.$

③a $\int\sqrt{r^2-x^2-y^2}\,dy=\int\sqrt{\left(\sqrt{r^2-x^2}\right)^2-y^2}\,dy$

$$=\int\sqrt{r^2-x^2}\cos(\theta)\cdot\sqrt{r^2-x^2}\cos(\theta)\,d\theta$$



$\rightarrow y=\sqrt{r^2-x^2}\sin(\theta)$

$dy=\sqrt{r^2-x^2}\cos(\theta)\,d\theta$

$\sqrt{r^2-x^2-y^2}=\sqrt{r^2-x^2}\cos(\theta)$

$$=(r^2-x^2)\int\cos^2(\theta)\,d\theta$$

$$=(r^2-x^2)\,\frac{1}{2}\int\left(1+\cos(2\theta)\right)d\theta$$

$$=\frac{1}{2}(r^2-x^2)\left[\theta+\frac{1}{2}\sin(2\theta)\right]+C$$

$$=\frac{1}{2}(r^2-x^2)\left(\theta+\cos(\theta)\sin(\theta)\right)+C$$

$$=\frac{1}{2}(r^2-x^2)\left(\arcsin\left(\frac{y}{\sqrt{r^2-x^2}}\right)+\frac{\sqrt{r^2-x^2-y^2}}{\sqrt{r^2-x^2}}\cdot\frac{y}{\sqrt{r^2-x^2}}\right)+C$$

$$=\frac{1}{2}(r^2-x^2)\left(\frac{y\sqrt{r^2-x^2-y^2}}{(r^2-x^2)}+\arcsin\left(\frac{y}{\sqrt{r^2-x^2}}\right)\right)+C$$

$$\therefore \int_{y=-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \sqrt{r^2-x^2-y^2}\, dy$$

$$= \frac{1}{2}(r^2-x^2)\left[\frac{y\sqrt{r^2-x^2-y^2}}{(r^2-x^2)} + \arcsin\left(\frac{y}{\sqrt{r^2-x^2}}\right)\right]_{y=-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}$$

$$= \frac{1}{2}(r^2-x^2)\Big(\big(0+\arcsin(1)\big) - \big(0+\arcsin(-1)\big)\Big)$$

$$= \frac{1}{2}(r^2-x^2)\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2}(r^2-x^2)$$

(3b) $\iint_R \sqrt{r^2-x^2-y^2}\, dA$

$$= \int_{x=-r}^{r} \frac{\pi}{2}(r^2-x^2)\, dx$$

$$= \frac{\pi}{2}\left[r^2x - \frac{1}{3}x^3\right]_{x=-r}^{r}$$

$$= \frac{\pi}{2}\left(\left(r^2\cdot r - \frac{r^3}{3}\right) - \left(r^2\cdot -r - \frac{(-r)^3}{3}\right)\right)$$

$$= \frac{\pi}{2}\left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right)$$

$$= \frac{\pi}{2} r^3\left(2 - \frac{2}{3}\right) = \frac{\pi}{2}\,\frac{4}{3}\, r^3$$

(4) Hence Vol(sphere) = 2 Vol(upper hemisphere)

$$= 2\iint_R \sqrt{r^2-x^2-y^2}\, dA$$

$$= 2\left(\frac{\pi}{2}\,\frac{4}{3}\, r^3\right)$$

$$= \frac{4}{3}\pi r^3$$

NB: We'll make this computation again later via two more methods to check our work. :)

Prop (Properties of Double Integrals):

Suppose $c \in \mathbb{R}$ and $f, g : R \subseteq \mathbb{R}^2 \to \mathbb{R}$.

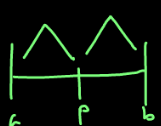
① $\iint_R (f+g)\, dA = \iint_R f\, dA + \iint_R g\, dA$

② $\iint_R (cf)\, dA = c \iint_R f\, dA$

} linearity of integral

③ If $f \leq g$ on R, then $\iint_R f\, dA \leq \iint_R g\, dA$.

④ If R can be split into two "nice" regions S and T which intersect only in their boundaries, then



$$\iint_R f\, dA = \iint_S f\, dA + \iint_T f\, dA.$$

⑤ $\text{Area}(R) = \iint_R 1\, dA.$

Cor: If $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, then we have $m\,\text{Area}(R) \leq \iint_R f(x,y)\, dA \leq M\,\text{Area}(R)$.

NB: These properties make computation possible for a much larger collection of regions than before!

MAJOR POINT: Sometimes it is easier to compute a double integral over a complicated region by splitting the region into several smaller regions. ☆

Ex: Compute the area of the triangle with vertices $(0,0)$, $(4,1)$, and $(7,3)$.

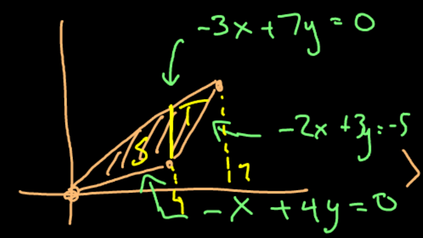
Sol: $\text{Area}(R) = \iint_R 1\,dA = \iint_S 1\,dA + \iint_T 1\,dA$

Lines $y = \frac{1}{3}(2x-5)$ and $y = \frac{1}{4}x$ intersect when $\frac{1}{3}(2x-5) = \frac{1}{4}x$

$\Leftrightarrow\ 8x - 20 = 3x$

$\Leftrightarrow\ x = 4$

Picture



$\vec{n}_1 = \langle -3, 7\rangle \perp \langle 7, 3\rangle$

$\vec{n}_2 = \langle -2, 3\rangle \perp \langle 3, 2\rangle$

$l_2 : -2(x-4) + 3(y-1) = 0$

i.e. $-2x + 3y = -5$

$\vec{n}_3 = \langle -1, 4\rangle \perp \langle 4, 1\rangle$

① $\iint_S 1\,dA$

$S = \{(x,y) : \frac{1}{4}x \le y \le \frac{3}{7}x,\ 0 \le x \le 4\}$

$\therefore \iint_S 1\,dA = \int_{x=0}^{4}\int_{y=\frac{1}{4}x}^{\frac{3}{7}x} 1\,dy\,dx = \int_{x=0}^{4} [y]_{y=1/4x}^{\frac{3}{7}x}\,dx$

$= \int_{x=0}^{4} \left(\frac{3}{7} - \frac{1}{4}\right)x\,dx = \left(\frac{3}{7} - \frac{1}{4}\right)\cdot \frac{1}{2}\left[x^2\right]_{x=0}^{4}$

$= \left(\frac{3}{7} - \frac{1}{4}\right)\cdot\frac{1}{2}(16 - 0) = 8\left(\frac{3}{7} - \frac{1}{4}\right) = \frac{24}{7} - 2 = \frac{10}{7}$

② $\iint_T 1\,dA = \int_{x=4}^{7} - \int_{y=\frac{3}{7}x}^{\frac{1}{3}(2x-5)} 1\,dy\,dx$ $\quad \{ T = \{(x,y) : 4 \le x \le 7,\ \frac{3}{7}x \ge y \ge \frac{1}{3}(2x-5)\}$

$= -\int_{x=4}^{7} [y]_{y=\frac{3}{7}x}^{\frac{1}{3}(2x-5)}\,dx = -\int_{x=4}^{7}\left(\frac{2}{3}x - \frac{5}{3} - \frac{3}{7}x\right)dx$

$= -\int_{x=4}^{7}\left(\left(\frac{2}{3} - \frac{3}{7}\right)x - \frac{5}{3}\right)dx = -\left[\frac{1}{2}\cdot\frac{5}{21}x^2 - \frac{5}{3}x\right]_{x=4}^{7}$

$= -\left(\left(\frac{1}{2}\cdot\frac{5}{21}\cdot 49 - \frac{5}{3}\cdot 7\right) - \left(\frac{1}{2}\cdot\frac{5}{21}\cdot 16 - \frac{5}{3}\cdot 4\right)\right) = -\left(\frac{35}{6} - \frac{35}{3} - \left(\frac{40}{21} - \frac{20}{3}\right)\right)$

$= -\left(-\frac{35}{6} - \left(\frac{40}{21} - \frac{140}{21}\right)\right) = -\left(\frac{100}{21} - \frac{35}{6}\right) = \frac{245 - 200}{21\cdot 2} = \frac{45}{21\cdot 2} = \frac{15}{14}$

③ $\text{Area}(R) = \iint_R 1\,dA = \iint_S 1\,dA + \iint_T 1\,dA = \frac{10}{7} + \frac{15}{14} = \frac{35}{14} = \frac{5}{2}$ //

Ex: Give bounds on $\iint_R \sqrt{9-x^2-y^2}\, dA$ for $R = \{(x,y) : x^2+y^2 \leq 4, x \geq 0\}$.

Sol: Note $5 \leq 9-x^2-y^2 \leq 9$ for all $(x,y) \in R$, so

$$\sqrt{5} \leq \sqrt{9-x^2-y^2} \leq \sqrt{9} = 3 \text{ for all } (x,y) \in R.$$

Moreover, R is the upper semicircle of radius 2, and hence has $\text{area}(R) = \frac{1}{2} \text{Area}(\text{circle}(\text{rad}=2)) = \frac{1}{2}\pi \cdot 2^2 = 2\pi$.

Hence we obtain estimates as follows

$$2\pi\sqrt{5} \leq \iint_R \sqrt{9-x^2-y^2}\, dA \leq 2\pi \cdot 3 = 6\pi$$

Ex: Compute the volume of the solid bounded by the cylinders $x^2+y^2=r^2$ and $y^2+z^2=r^2$ for $r>0$.

Sol: Left to you (Hint: Draw a picture ...).